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Exact Solutions in Barker's Homogeneous Isotropic Cosmologies

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We derive a complete set of new exact $k = 0, \pm 1$ radiation solutions of Barker's homogeneous isotropic cosmologies. In the very early universe they reduce to the asymptotic solutions of Yepes and Domínguez-Tenreiro. Consistency with the standard cosmological model constrains the solution's free parameters.

1. INTRODUCTION

In the Brans–Dicke (1961) scalar–tensor theory of gravity Newton's gravitational constant G is time-dependent and a dynamical scalar field, with dimensions of G^{-1}, is present. But in Barker's (1978) theory, a special case of the Nordtvedt (1970) formulation of the scalar–tensor theory (Bergmann, 1968; Wagoner, 1970; Will, 1974), G is a genuine constant and the scalar field is dimensionless. Observations appear to favor a constant G (Norman, 1986).

Exact analytic solutions of Barker's homogeneous isotropic cosmologies are scarce. They have been obtained by Barker (1978) for an empty, flat ($k = 0$) universe and by Lorenz-Petzold (1984) for empty, nonflat ($k = \pm 1$) models. No other exact solution describing a homogeneous and isotropic but more realistic universe is found in the literature. Approximate and numerical solutions have been discussed by Yepes and Domínguez-Tenreiro (1986). Conformally flat static vacuum solutions and Bianchi type VI_0 solutions were investigated by Shanthi (1989) and Shanthi and Rao (1990).

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In this paper we use the method of Lorenz-Petzold (1984) to solve exactly the Barker–Robertson–Walker (BRW) field equations for the early radiation-dominated universe. Consistency between the thermal history of the very early universe in the resulting and standard cosmology can be established once the solution's free parameters are suitably constrained.

In Section 2 we display the BRW field equations. In Section 3 we solve them for the radiation universe. Section 4 is devoted to a comparison with the standard model. Finally, in Section 5 we summarize and discuss our results.

2. BRW FIELD EQUATIONS

The field equations of the Nordtvedt (1970) general scalar–tensor theory are

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^{\sigma}{}_{\sigma}=-\frac{8\pi\bar{G}}{\phi}T_{\mu\nu}-\frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right)-\phi^{-1}(\phi_{;\mu\nu}-g_{\mu\nu}\,\Box\phi) \tag{1}$$

$$\Box\phi=\frac{1}{3+2\omega}\left(8\pi\bar{G}T^{\sigma}{}_{\sigma}-\phi_{,\lambda}\phi^{,\lambda}\frac{d\omega}{d\phi}\right) \tag{2}$$

In these equations $\bar{G}$ is an arbitrary *constant* related to G by

$$G=\frac{\bar{G}}{\phi}\left(\frac{4+2\omega}{3+2\omega}\right) \tag{3}$$

Equations (1) and (2) imply

$$T^{\mu\nu}{}_{;\nu}=0 \tag{4}$$

i.e., the matter energy-momentum tensor is conserved.

Barker's (1978) gravity emerges on setting $\bar{G}=G$, so that $\omega=(4-3\phi)/(2\phi-2)$ and ϕ is dimensionless. With the Robertson–Walker (RW) metric and the perfect fluid form of the energy-momentum tensor the preceding field equations can then be written as [$\alpha^{-1}=(8/3)\pi G$ and $(\cdot)^{\cdot}\equiv(d/dt)(\cdot)$]

$$(\ln R)^{\cdot\cdot}+[(\ln R)^{\cdot}]^2-(\ln R)^{\cdot}(\ln\phi)^{\cdot}+\frac{8-3\phi}{12(\phi-1)}[(\ln\phi)^{\cdot}]^2=\frac{1}{2}\alpha^{-1}\phi^{-1}R^2[2(\rho-6p)-3\phi(\rho-3p)] \tag{5}$$

$$[(\ln R)^{\cdot}]^2+(\ln R)^{\cdot}(\ln \phi)^{\cdot}-\frac{4-3\phi}{12(\phi-1)}[(\ln \phi)^{\cdot}]^2+kR^{-2}=\alpha^{-1}\phi^{-1}\rho \quad (6)$$

$$\ddot{\phi}+3(\ln R)^{\cdot}\dot{\phi}-\frac{\dot{\phi}^2}{2(\phi-1)}=3\alpha^{-1}(\phi-1)(\rho-3p) \quad (7)$$

where R is the RW scale factor and p and ρ are the cosmic pressure and energy density, respectively. Equations (5)–(7) imply

$$\frac{d}{dt}(\rho R^3)=-3pR^2\dot{R} \quad (8)$$

which is a relation valid in any metric theory of gravity with a conserved perfect-fluid energy-momentum tensor.

3. RADIATION SOLUTIONS

We consider the early universe with the radiation equation of state $p=\frac{1}{3}\rho$. Then equation (8) gives $\rho=\gamma/R^4$, where γ is a positive constant. Introducing the Lorenz-Petzold (1984) variables g and η defined by $g=R^2\phi$ and $dt=R\,d\eta$, we note that equations (5)–(7) can now be decoupled to yield

$$g''+4kg=2\alpha^{-1}\gamma \quad (9)$$

$$(g')^2+4kg^2=4\alpha^{-1}\gamma g+\tfrac{1}{3}c^2 \quad (10)$$

$$\frac{(\phi')^2}{(\phi-1)\phi^2}=\frac{c^2}{g^2} \quad (11)$$

where a prime denotes differentiation with respect to η and c^2 is a constant ($c^2\equiv 0$ or, equivalently, $\phi\equiv$ const corresponds to general relativity and is therefore excluded).

The general solutions that follow from equations (9) and (10) are as follows:

(i) $k=0$:

$$g=\alpha^{-1}\gamma\eta^2+a\eta+b \quad (12)$$

where a and b are integration constants satisfying

$$a^2=4\alpha^{-1}\gamma b+\tfrac{1}{3}c^2 \quad (13)$$

(ii) $k=1$:

$$g=a\sin(2\eta+b)+\tfrac{1}{2}\alpha^{-1}\gamma \quad (14)$$

where

$$4a^2 = \alpha^{-2}\gamma^2 + \tfrac{1}{3}c^2 \tag{15}$$

(iii) $k = -1$:

$$g = a\sinh(2\eta + b) - \tfrac{1}{2}\alpha^{-1}\gamma \tag{16}$$

where

$$4a^2 = -\alpha^{-2}\gamma^2 + \tfrac{1}{3}c^2 \tag{17}$$

Some general characteristics of the solutions of equation (11) that do not depend on k are the following. For $c^2 > 0$, $\phi > 1$. Hence ϕ falls on branch #1 of the hyperbola (Barker, 1978), $\omega = (4 - 3\phi)/(2\phi - 2)$, where (ω, ϕ) runs from $(-3/2, \infty)$ at $t = 0$ to $(\infty, 1)$ as $t \to \infty$. Then, assuming $\eta \to 0$ as $t \to 0$, we have $\phi' < 0$ and $g = R^2\phi \geq 0$ with the equality, if valid, holding at $\eta = 0$. On the other hand, $c^2 < 0$ implies $\phi < 1$, so that ϕ is on branch #2 of the ω–ϕ hyperbola (Barker, 1978), where (ω, ϕ) goes from $(-3/2, -\infty)$ to $(-\infty, 1)$ and $\phi' > 0$. In this case $\lim_{\eta \to 0} g \leq 0$. With these remarks in mind, the solutions of equation (1) are conveniently discussed separately as follows.

3.1. Flat Universe ($k = 0$)

$c^2 > 0$: Then $g \to b \geq 0$ as $t \to 0$. (We admit, *a priori*, $b = 0$, but will shortly show that it should be excluded.) With $b > 0$ equations (11)–(13) give

$$\phi = \operatorname{cosec}^2 \frac{\sqrt{3}}{2} \ln\left|\frac{1 + 2\sqrt{3}\gamma\alpha^{-1}(\sqrt{3}a + |c|)^{-1}\eta}{1 + 2\sqrt{3}\gamma\alpha^{-1}(\sqrt{3}a - |c|)^{-1}\eta}\right| \tag{18}$$

But if $b = 0$, we have, up to an additive integration constant,

$$2a\tan^{-1}(\phi - 1)^{1/2} = |c|\ln|(\eta + \alpha\gamma^{-1}a)\eta^{-1}| \tag{19}$$

so that as $\eta \to 0$ the principal value of the left-hand side approaches πa, whereas the right-hand side becomes infinite. Because of this pathological feature we exclude $b = 0$.

$c^2 < 0$: Here $g \to b \leq 0$ as $\eta \to 0$, so that $a^2 < 0$. But $a^2 < 0$ is inconsistent with a real ϕ. Hence $c^2 < 0$ is inadmissible.

3.2. Closed Universe ($k = 1$)

$c^2 > 0$: In this case the ϕ solution is

$$\phi = \operatorname{cosec}^2 \frac{\sqrt{3}}{2} \ln \Gamma\left|\frac{1 + \sqrt{3}\alpha^{-1}\gamma(2\sqrt{3}a + |c|)^{-1}\tan(\eta + b/2)}{1 + \sqrt{3}\alpha^{-1}\gamma(2\sqrt{3}a - |c|)^{-1}\tan(\eta + b/2)}\right| \tag{20}$$

where

$$\Gamma = \left| \frac{1+\sqrt{3}\alpha^{-1}\gamma(2\sqrt{3}a - |c|)^{-1}\tan(b/2)}{1+\sqrt{3}\alpha^{-1}\gamma(2\sqrt{3}a + |c|)^{-1}\tan(b/2)} \right| \tag{21}$$

In particular, for $b = 0$, $\Gamma = 1$ and

$$\phi = \operatorname{cosec}^2 \frac{\sqrt{3}}{2} \ln \left| \frac{1+\sqrt{3}\alpha^{-1}\gamma(2\sqrt{3}a + |c|)^{-1}\tan\eta}{1+\sqrt{3}\alpha^{-1}\gamma(2\sqrt{3}a - |c|)^{-1}\tan\eta} \right| \tag{22}$$

$c^2 < 0$: Here $g \to_{\eta \to 0} a \sin b + \frac{1}{2}\alpha^{-1}\gamma \leq 0$, so that if b is real, $\alpha^{-2}\gamma^2 \leq 4a^2 \sin^2 b \leq 4a^2$, implying by equation (15) the contradictory result $c^2 \geq 0$. Hence solutions with $c^2 < 0$ do not exist.

3.3. Open Universe ($k = -1$)

$c^2 > 0$: $g \to_{\eta \to 0} a \sinh b - \frac{1}{2}\alpha^{-1}\gamma$. So, provided $|b| \geq \sinh^{-1}(\gamma/2\alpha|a|)$, where $b/a > 0$, we find

$$\phi = \operatorname{cosec}^2 \frac{\sqrt{3}}{2} \ln \left| \frac{1+\Gamma_+ \tanh\eta}{1+\Gamma_- \tanh\eta} \right| \tag{23}$$

where

$$\Gamma_\pm = \frac{(2\sqrt{3}a \pm |c|)\tanh(b/2) + \sqrt{3}\alpha^{-1}\gamma}{2\sqrt{3}a \pm |c| + \sqrt{3}\alpha^{-1}\gamma \tanh(b/2)} \tag{24}$$

$c^2 < 0$: In this case equation (17) implies $a^2 < 0$. Hence if b is real, ϕ is complex, which is excluded.

4. STANDARD MODEL CONSTRAINTS

We consider now the forms of the previous solutions when $\phi \gg 1$. As we shall see, the obtained results are consistent with the standard model provided the solution parameters are appropriately constrained.

4.1. $k = 0$

For $\phi \gg 1$, or equivalently, $\eta \to 0$, equation (18) reduces to

$$\phi = \frac{(3a^2 - c^2)^2 \alpha^2 \eta^{-2}}{36\gamma^2 c^2} = \frac{4b^2}{c^2}\eta^{-2} \tag{25}$$

and $R^2\phi = g \to b > 0$, so that

$$R = \frac{|c|}{2\sqrt{b}}\eta = \left(\frac{c^2}{b}\right)^{1/4} t^{1/2} \tag{26}$$

Thus

$$\phi = \frac{b^{3/2}}{|c|} t^{-1} \tag{27}$$

Also, in terms of the temperature T,

$$\gamma R^{-4} = \rho = \frac{\pi^2}{30} g_{\text{eff}} T^4 \tag{28}$$

where g_{eff} is the effective number of spin degrees, which we assume to be constant in the very early universe. One therefore has from (26) the temperature–time relation

$$T = \left(\frac{30\gamma b}{\pi^2 c^2 g_{\text{eff}}}\right)^{1/4} t^{-1/2} \tag{29}$$

The standard scenario of primordial nucleosynthesis is one of standard cosmology's principal successes. Its bases are the formula $R = (4\alpha^{-1}\gamma)^{1/4} t^{1/2}$ for the dependence of R upon t and the temperature–time relation that then follows from equation (28). Multiplicative factors apart, equations (26) and (29) have the standard model time dependence. In order to preserve the standard nucleosynthesis description, we identify equation (29) with the standard relation

$$T = \left(\frac{15\alpha}{2\pi^2 g_{\text{eff}}}\right)^{1/4} t^{-1/2} \tag{30}$$

Hence $c^2 = 4b\gamma\alpha^{-1}$, $R = (4\alpha^{-1}\gamma)^{1/4} t^{1/2}$, and using equation (13), $\sqrt{3}a = \pm 2|c|$. To prevent ϕ in equation (18) from developing a pole at a finite $\eta \neq 0$, we select $\sqrt{3}a = +2|c|$. Then, from (18),

$$\phi = gR^{-2} = \operatorname{cosec}^2 \frac{\sqrt{3}}{2} \ln\left|\frac{3|c| + 2\sqrt{3}\gamma\alpha^{-1}\eta}{3|c| + 6\sqrt{3}\gamma\alpha^{-1}\eta}\right| \tag{31}$$

4.2. $k = 1$

In the limit $\eta \to 0$, equation (20) becomes

$$\phi = \frac{(\alpha^{-1}\gamma + 2a \sin b)^2 \eta^{-2}}{c^2} \tag{32}$$

and $R^2\phi = g \to a \sin b + \frac{1}{2}\alpha^{-1}\gamma$, so that

$$R = \left(\frac{2c^2}{\alpha^{-1}\gamma + 2a \sin b}\right)^{1/4} t^{1/2} \tag{33}$$

In terms of t, equation (32) reads

$$\phi = \frac{(\alpha^{-1}\gamma + 2a \sin b)^{3/2}}{2\sqrt{2}\,|c|} t^{-1} \tag{34}$$

The temperature–time relation in this case is

$$T = \left(\frac{15\gamma^2(1 + 2\alpha a\gamma^{-1} \sin b)}{\pi^2 g_{\text{eff}} \alpha c^2}\right)^{1/4} t^{-1/2} \tag{35}$$

In these equations a is given in terms of c by equation (15).

Identification of equation (35) with the standard temperature–time relation yields

$$2a \equiv \left(\alpha^{-2}\gamma^2 + \frac{c^2}{3}\right)^{1/2} = \frac{\alpha^{-1}\gamma \sin b}{3}\left[1 + \left(1 + \frac{15}{\sin^2 b}\right)^{1/2}\right] \tag{36}$$

In particular,

$$2a > \tfrac{5}{3}\alpha^{-1}\gamma \sin b, \qquad b \neq 0, \pi/2 \tag{37a}$$

$$2a = \tfrac{5}{3}\alpha^{-1}\gamma, \qquad b = \pi/2 \tag{37b}$$

$$2\sqrt{3}a = \sqrt{5}\alpha^{-1}\gamma, \qquad b = 0 \tag{37c}$$

In the last case equation (22) becomes

$$\phi = gR^{-2} = \text{cosec}^2 \frac{\sqrt{3}}{2} \ln\left|\frac{1 + (1/\sqrt{3})(\sqrt{5} - \sqrt{2}) \tan \eta}{1 + (1/\sqrt{3})(\sqrt{5} + \sqrt{2}) \tan \eta}\right| \tag{38}$$

4.3. $k = -1$

The asymptotic $\eta \to 0$ form of equation (23) is

$$\phi = D\eta^{-2} \tag{39}$$

where

$$c^2 D = (2a \sinh b - \alpha^{-1}\gamma)^2 \tag{40}$$

But $R^2\phi = g \to a \sinh b - \frac{1}{2}\alpha^{-1}\gamma$ as $\eta \to 0$. Hence

$$R = \left(\frac{2|c|}{|D|^{1/2}}\right)^{1/4} t^{1/2} \tag{41}$$

and

$$\phi = \left(\frac{c^2 D^3}{64}\right)^{1/4} t^{-1} \tag{42}$$

The temperature–time relation is

$$T = \left(\frac{15\gamma |D|^{1/2}}{\pi^2 g_{\text{eff}} |c|} \right)^{1/4} t^{-1/2} \tag{43}$$

so that the thermal history of the very early universe agrees with the standard one if

$$\tfrac{1}{2}\alpha\gamma^{-1}c^2 = 2a \sinh b - \alpha^{-1}\gamma \tag{44}$$

When c^2 in this equation is eliminated using equation (17) a quadratic equation in a is obtained. It implies that $\sinh^2 b \geq 15$ with the equality corresponding to $a = \frac{1}{6}\alpha^{-1}\gamma \sinh b = \pm(\sqrt{15}/6)\alpha^{-1}\gamma$. In this case $c^2 = 8\alpha^{-2}\gamma^2$ and the coefficients $\Gamma_\pm$ in equation (24) become

$$\Gamma_+ = \frac{\sqrt{15} \pm \sqrt{6}}{\sqrt{10} \pm 4} \tag{45a}$$

$$\Gamma_- = \frac{\sqrt{15} \mp \sqrt{6}}{-\sqrt{10} \pm 4} \tag{45b}$$

Of the exhibited sign combinations, those that produce a zero or a pole of the argument of the logarithm in equation (23) should be excluded.

5. SUMMARY AND DISCUSSION OF RESULTS

We have derived in this paper new exact solutions to Barker's homogeneous isotropic cosmologies. These solutions are relevant to the early radiation era of flat, closed, and open universes. They do not reduce to the vacuum solutions of Barker (1978) and Lorenz-Petzold (1984) in the limit of a vanishing radiation density.

The t-behavior of the solutions as $t \to 0$ [equations (26), (27), (33), (34), (41), and (42)] is curvature-independent and agrees with the approximate results of Yepes and Domínguez-Tenreiro (1986) [see their equations (44)–(52)]. The scale factor then grows like $t^{1/2}$, just as in the standard model. It is therefore reasonable to require consistency between the thermal histories of the very early universe here and in standard cosmology. The motivation is to preserve the apparently successful standard nucleosynthesis scenario. This requirement yields several constraints on the solution's free parameters. The validity of the asymptotic forms of the solutions at the time of nucleosynthesis can be checked as follows.

With the aforementioned standard model constraints on the solution's parameters, η is given asymptotically ($\forall k$) by $\eta = (t/\tau)^{1/2}$, where $\tau = (4\alpha\gamma^{-1})^{-1/2}$. Now one can set $\gamma = \rho_{r\text{p}} R_\text{p}^4$, where the subscript "p" denotes present-day values of the parameters. Taking $\rho_{r\text{p}} \approx 10^{-51}\,(\text{GeV})^4$ and $R_\text{p} \approx 10^{43}\,(\text{GeV})^{-1}$, we find $\tau \approx 2.5 \times 10^{17}$ sec, of the order of the age of the universe. Thus at the beginning of nucleosynthesis ($t \approx 1$ sec) we have $\eta \approx 10^{-9}$, so that the small-η approximation is well justified. In fact, $\eta \ll 1$ throughout the standard radiation era extending to $t \approx 10^{12}$ sec (where $\eta \approx 10^{-3}$). Hence the early universe in Barker's cosmologies evolves essentially as in the standard model. The scalar field itself assumes a particularly simple form when $k = 1$, $b = 0$, and η small. It is, from equation (38), $\phi = (2\eta^2)^{-1} = \tau/(2t) \approx 10^{17}$ at $t \approx 1$ sec. Then $\omega \approx -3/2$.

As noted by Yepes and Domínguez-Tenreiro (1986), equations (26), (33), and (41) imply that the horizon problem survives in Barker's models. But the presence of a scalar field might provide a mechanism for its solution. Extended inflation in scalar–tensor theories with a dimensional scalar field has been proposed recently in order to resolve the cosmological problems that plague the standard model (Mathiazhagan and Johri, 1984; La and Steinhardt, 1989*a,b*; Garcia-Bellido and Quirós, 1990). However, in Brans–Dicke cosmology, inflation appears to require a Brans–Dicke constant parameter $\omega \leq 30$ (Garcia–Bellido and Quirós, 1990; La *et al.*, 1989; Weinberg, 1989). Such a bound is ruled out by light deflection and time-delay experiments, which constrain ω to be ≥ 500 (Will, 1984). In Barker's cosmology, on the other hand, ω varies and a small ω in the cosmic dawn is compatible with $\omega \geq 500$ today.

Finally, we remark that although the physical meaning of a *dimensionless* ϕ is not clear *a priori*, it has been shown (Sáez, 1987) that such a field can be physically admissible.

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NOTE ADDED IN PROOF

After this work was submitted for publication J. D. Barrow published a paper (*Physical Review D*, **47**, 5329 (1993)) in which similar exact solutions of scalar–tensor cosmologies were discussed.

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